Monge's Method:

1. Let the differential equation be

$$Rr + Ss + Tt = V \quad \rightarrow (1)$$

where R, S, T and V are functions of x, y, z, p & q.

$$r = \frac{\partial p}{\partial x}, \quad s = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}, \quad t = \frac{\partial q}{\partial y}.$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = r\,dx + s\,dy$$

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = s\,dx + t\,dy$$

$$\Rightarrow r = \frac{dp - s\,dy}{dx}, \quad t = \frac{dq - s\,dx}{dy}$$

$$(1) \Rightarrow R\left(\frac{dp - s\,dy}{dx}\right) + Ss + T\left(\frac{dq - s\,dx}{dy}\right) = V$$

$$\Rightarrow R(dp\,dy - s(dy)^2) + Ss\,dx\,dy + T(dq\,dx - s(dx)^2) = V\,dx\,dy$$

$$\Rightarrow (R\,dp\,dy + T\,dq\,dx - V\,dx\,dy) + s(S\,dx\,dy - R(dy)^2 - T(dx)^2) = 0$$

$\therefore$ Monge's subsidiary equations are

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \quad \rightarrow (2)$$

$$R(dy)^2 + T(dx)^2 - S\,dx\,dy = 0 \quad \rightarrow (3)$$

Suppose (3) can be factored into 2 linear factors $dy - m_1 dx = 0$, $dy - m_2 dx = 0$

These give us two integrals $u_1 = a$, $v_1 = b$

Put $dy = m_1 dx$ in (2)

$$\Rightarrow m_1 R\,dp\,dx + T\,dq\,dx - m_1 V(dx)^2 = 0$$

$$\Rightarrow m_1 R\,dp + T\,dq - m_1 V\,dx = 0$$

On integrating this will give an intermediate integral. Similarly $dy = m_2 dx$ gives another intermediate integral.

On solving these two integrals ~~to~~ get

p and q in terms of x and y.
Substitute for p and q in
$dz = p\,dx + q\,dy$
On integrating this we get the solution.

Ex $r = a^2 t$.

$$r = \frac{dp - s\,dy}{dx}, \quad t = \frac{dq - s\,dx}{dy}$$

$$\Rightarrow \quad dp\,dy - s(dy)^2 = a^2 (dq\,dx - s(dx)^2)$$

$$\Rightarrow \quad dp\,dy - a^2 dq\,dx - s\left((dy)^2 - a^2 (dx)^2\right) = 0$$

$\therefore$ Monge's subsidiary equations are

$$dp\,dy - a^2 dq\,dx = 0 \rightarrow (a)$$

$$\& \quad (dy)^2 - a^2 (dx)^2 = 0 \rightarrow (b)$$

$(b) \Rightarrow \quad dy = \pm a\,dx \quad \Rightarrow y = ax + c_1$
$\& \; y = -ax + c_2$

$dy = a\,dx \quad \Rightarrow \quad a\,dp - a^2 dq = 0$
$\Rightarrow p - aq = f_1(c_1) = f_1(y - ax)$
$\rightarrow (c)$

$dy = -a\,dx \quad \Rightarrow \quad -a\,dp - a^2 dq = 0 \Rightarrow p + aq = f_2(c_2)$
$= f_2(y + ax)$
$\rightarrow (d)$

$(c) \& (d) \Rightarrow \quad 2p = f_1(y - ax) + f_2(y + ax)$

$$2aq = f_2(y + ax) - f_1(y - ax)$$

$$\therefore 2dz = \left(f_1(y - ax) + f_2(y + ax)\right)dx + \frac{1}{a}\left(f_2(y + ax) - f_1(y - ax)\right)dy$$

$$= \frac{1}{a}\{f_1(y - ax)\; d(ax - y)\} + \frac{1}{a}\left(f_2(y + ax)\right) d(ax + y)$$

$$\Rightarrow 2z = -\frac{1}{a}\int f_1(y - ax)\, d(y - ax) + \frac{1}{a}\int f_2(y + ax)\, d(y + ax)$$

$$\Rightarrow z = \phi_1(y - ax) + \phi_2(y + ax)$$

② Let the pde be $Rr + Ss + Tt + U(rt - s^2) = V$ $\rightarrow$ ②

where R, S, T, U, V are functions of x, y, z, p & q.

As before, we have $r = \dfrac{dp - s\,dy}{dx}$ &

$$t = \frac{dq - s\,dx}{dy}$$

$$\Rightarrow \quad R(dp - s\,dy)\,dy + Ss\,dx\,dy + T(dq - s\,dx)\,dx$$
$$+ U\left(\frac{(dq - s\,dx)(dp - s\,dy)}{dx\,dy} - s^2\right) = V\,dx\,dy$$

$$\Rightarrow \left(R\,dp\,dy + T\,dq\,dx + U\,dp\,dq - V\,dx\,dy\right)$$
$$+ s\left(-R(dy)^2 + S\,dx\,dy - T(dx)^2 - U\,dp\,dx - U\,dq\,dy\right) = 0$$

$\therefore$ Monge's subsidiary equations are

$$A = R\,dp\,dy + T\,dq\,dx + U\,dp\,dq - V\,dx\,dy = 0$$

$$B = R(dy)^2 - S\,dx\,dy + T(dx)^2 + U\,dp\,dx + U\,dq\,dy = 0$$

In general, it is not easy to factorize these equations. So we introduce a multiplier. Let λ be an unknown multiplier such that the equation $B + \lambda A = 0$ can be factored into linear factors

$$A_1\,dy + B_1\,dx + C_1\,dp \quad \& \quad A_2\,dy + B_2\,dx + C_2\,dq$$

$$\therefore (B + \lambda A) = (A_1\,dy + B_1\,dx + C_1\,dp)(A_2\,dy + B_2\,dx + C_2\,dq) = 0$$

Equating the coefficients on both sides we have

$R = A_1 A_2$	$U = C_1 B_2 = A_1 C_2$
$-\lambda V - S = A_1 B_2 + B_1 A_2$	$\lambda R = C_1 A_2$
$T = B_1 B_2$	$\lambda T = B_1 C_2$
	$\lambda U = C_1 C_2$

Put $A_1 = R$, $A_2 = 1$, $B_1 = kT$, $B_2 = \frac{1}{k}$,
$C_1 = mU$ & $C_2 = \frac{\lambda}{m}$

$\Rightarrow$ $-\lambda V - S = A_1B_2 + B_1A_2 = kT + \frac{R}{k}$
$U = \lambda_1 C_2 = R\frac{\lambda}{m} = \frac{m}{k} U$

$\Rightarrow$ $m = k$. So $B_1C_2 = kT\frac{\lambda}{m} = \lambda T$ (True)
& $C_1A_2 = mU = \lambda R \Rightarrow m = k = \frac{\lambda R}{U}$

So $-\lambda V - S = \frac{\lambda R}{U} T + \frac{RU}{\lambda R}$

$$\Rightarrow -\lambda^2 V - \lambda S = \frac{\lambda^2 RT}{U} + U$$

$$\Rightarrow \lambda^2 (RT + UV) + \lambda S U + U^2 = 0 \rightarrow (\mathrm{I})$$

If λ is a root of this equation, then
$(B + \lambda A) = (R\,dy + kT\,dx + kU\,dp)(dy + \frac{1}{k}dx + \frac{\lambda}{k}dq) = 0$
As $k = \frac{\lambda R}{U}$

$$\Rightarrow \frac{R}{U}(U\,dy + \lambda T\,dx + \lambda U\,dp)\ \frac{1}{\lambda R}(\lambda R\,dy + U\,dx + \lambda U\,dq) = 0 \rightarrow (\mathrm{II})$$

$$\Rightarrow \left.\begin{aligned} U\,dy + \lambda T\,dx + \lambda U\,dp = 0 \\ \lambda R\,dy + U\,dx + \lambda U\,dq = 0 \end{aligned}\right\}$$

On integrating these two equations we get two intermediate integrals. We solve these two integrals for p & q, substitute in $dz = p\,dx + q\,dy$ and integrate to get the complete integral.

Ex $\quad 2r + e^x t - (rt - s^2) = 2e^x$

$R = 2$, $S = 0$, $T = e^x$, $U = -1$, $V = 2e^x$

$A = 2\,dp\,dy + e^x\,dq\,dx - dp\,dq - 2e^x\,dx\,dy$
$= 2dy\,(dp - e^x dx) + dq\,(e^x dx - dp)$
$= (2dy - dq)(dp - e^x dx) = 0$

$B = 2(dy)^2 - 0 + e^x (dx)^2 - dp\,dx - dq\,dy$
$= dy\,(2dy - dq) + dx\,(e^x dx - dp)$
$= 0$

We observe that we can factorise A directly

$2dy - dq = 0$ & $dp - e^x dx = 0$

$\Rightarrow q - 2y = a$ & $p - e^x = b$

With this $B = 0$ is automatically satisfied

$dz = p\,dx + q\,dy$

$= (b + e^x)dx + (a + 2y)dy$

$\Rightarrow z = bx + e^x + ay + y^2 + c.$

Ex $3r + 4s + t + (rt - s^2) = 1$

$\Rightarrow R = 3,\quad S = 4,\quad T = 1,\quad U = 1,\quad V = 1$

$A = 3\,dp\,dy + dq\,dx + dp\,dq - dx\,dy = 0$

$B = 3(dy)^2 - 4\,dx\,dy + (dx)^2 + dp\,dx + dq\,dy = 0$

$B + \lambda A = (A_1 dy + B_1 dx + C_1 dp)(A_2 dy + B_2 dx + C_2 dq)$

where $\lambda^2(3.1 + 1.1) + \lambda.4.1 + 1^2 = 0$

(I) $\Rightarrow 4\lambda^2 + 4\lambda + 1 = 0 \Rightarrow (2\lambda + 1)^2 = 0 \Rightarrow \lambda = -1/2$

(II) $\Rightarrow \frac{3}{1}\left(dy - \frac{1}{2}dx - \frac{1}{2}dp\right)\left(\frac{-2.1}{3}\right)\left(-\frac{3}{2}dy + dx - \frac{1}{2}dq\right) = 0$

$$\Rightarrow \begin{cases} 2dy - dx - dp = 0 & \Rightarrow p = 2y - x + a \\ \& \; 3dy - 2dx + dq = 0 & \Rightarrow q = 2x - 3y + b \end{cases}$$

$dz = (2y - x + a)dx + (2x - 3y + b)dy$

$= -x\,dx + a\,dx - 3y\,dy + b\,dy + 2\underbrace{(x\,dy + y\,dx)}_{d(xy)}$

$\Rightarrow z = -\frac{x^2}{2} + ax - 3\frac{y^2}{2} + by + 2xy + C$

$\longleftrightarrow$